Spatiotemporal models

Applied Spatial Statistics

Spatiotemporal data

- Say Y_i is observed at location $\mathbf{s}_i$ and time t_i
- Example: Y_i is the air pollution in Los Angeles ($\mathbf{s}_i$) on August 1, 2020 (t_i)
- We also have covariates, $\mathbf{X}_i = (X_{i1}, ..., X_{ip})$
- Example: $X_{i1} = 1$ is the year of t_i
- Example: $X_{i2} = 1$ is the elevation of $\mathbf{s}_i$
- Example: X_{i3} is the temperature at time t_i and location $\mathbf{s}_i$

Objectives

- Estimate covariate effects while accounting for spatiotemporal correlation
- Make predictions in space where there is no data
- Short-term forecasting
- Test for changes over time

General model

- As for spatial data, we decompose the data into a mean, correlated residuals and uncorrelated residuals

$$Y_i = \mu_i + Z_i + \varepsilon_i$$

- The mean is $\mu_i = \beta_0 + \sum_{j=1} X_{ij}\beta_j$

- Z_i is correlated across space and time

- The nugget is $\varepsilon_i \sim \text{Normal}(0, \tau^2)$, independent over i

Discrete versus continuous time

- Models for Z_i are different if time is discrete or continuous
- Discrete time: $t_i \in \{1, 2, 3, ...\}$, such as t_i is a year
- Continuous time: $t_i \in \mathcal{R}$, such as t_i is the time of a measurement to the millisecond
- Modeling also depends on whether the same spatial locations are measured over time or not
- There are countless models, but we will cover spatial autoregressive models for discrete time and separable models for continuous time[1]

[1] Both methods can actually be applied to both cases

Time-series data

- First, let's review times series analysis
- Let Y_t be the measurement at time t (discrete time)
- Say $Y_t = \mu_t + Z_t + \varepsilon_t$
- Assume the covariance of Z_t is stationary, i.e., the same over time
- Then $\text{Cov}(Z_t, Z_{t+h}) = \sigma^2\psi(h)$ where ψ is the autocorrelation function

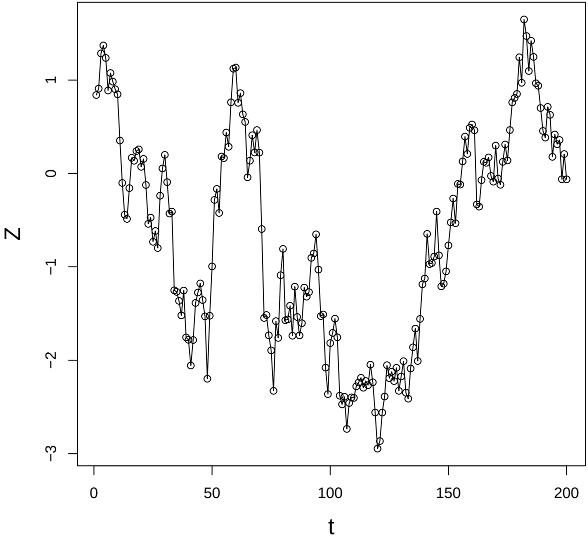
Autoregressive (AR) model

- There are many autocorrelation functions
- The simplest model is the AR1 model

$$
\begin{aligned}
Z_1 &\sim \text{Normal}(0, \sigma^2) \\
Z_t | Z_{t-1} &\sim \text{Normal}\{\varphi Z_{t-1}, (1-\varphi^2)\sigma^2\}
\end{aligned}
$$

- Under this model, $\text{Var}(Z_t) = \sigma^2$ for all t
- The ACF is $\psi(h) = \varphi^h$ for $\varphi \in (-1, 1)$

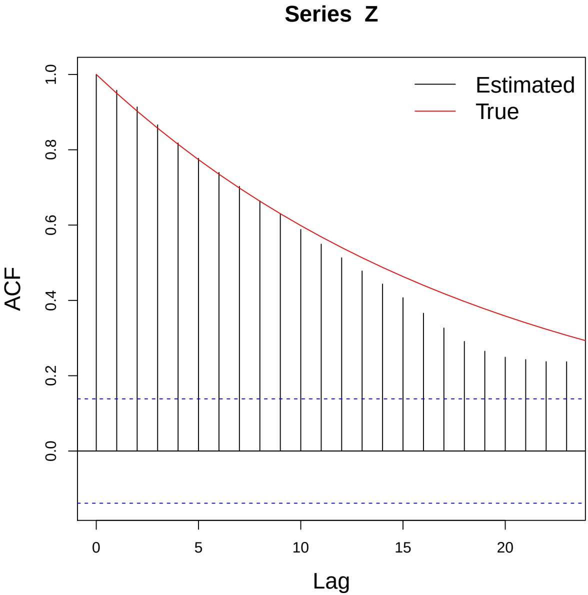
Random samples with $\varphi = 0.95$



Autocorrelation function (ACF)

- The true autocorrelation function at lag h is $\psi(h)$
- The sample ACF is computed like the sample variogram
- This plays a similar role to the variogram
- It is used mostly for exploratory data analysis
- The next slide plots the sample and true ACF φ^h (red)

ACF of the random sample



Spatiotemporal exploratory analysis

- Do a times series analysis at each site: check the mean structure and ACF
- Do a separate spatial analysis at each time point: check mean structure and the variogram
- Plot the time series results over space and the spatial results over time
- Check for differences over space and time

Spatiotemporal random effects model

- The overall model is $Y_i = \mu_i + Z_i + \varepsilon_i$
- Assume data are available at m time points $t_i \in \{1, ..., m\}$
- For each time point there is a sample at the same n location $\mathbf{s}_i \in \{S_1, ..., S_n\}$
- Let $Z(S, T)$ be Z_i for the observation with $\mathbf{s}_i = S$ and $t_i = T$

Spatiotemporal random effects model

- The spatiotemporal term Z_i can be split into spatial and temporal random effects $Z(S, T) = \alpha(S) + \gamma(T)$
- The spatial terms have $\text{Var}\{\alpha(S)\} = \sigma_S^2$ and spatial correlation
- The temporal terms have $\text{Var}\{\gamma(T)\} = \sigma_T^2$ and autocorrelation
- The total variance is $\text{Var}(Y_i) = \tau^2 + \sigma_S^2 + \sigma_T^2$
- The proportion of variance explained by spatial and temporal variation are

$$\frac{\sigma_S^2}{\tau^2 + \sigma_S^2 + \sigma_T^2} \quad \text{and} \quad \frac{\sigma_T^2}{\tau^2 + \sigma_S^2 + \sigma_T^2}$$

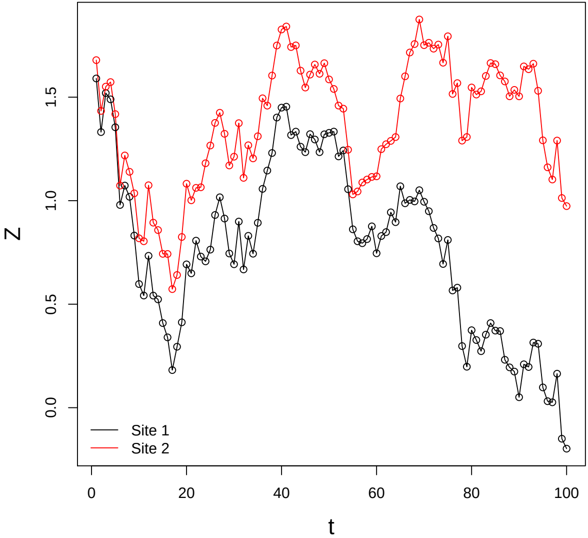
Spatial autoregressive model

- The previous model has the same spatial trend for each time, and the same time trend for each location
- A more flexible model may be needed
- A spatial AR1 model begins with the AR1 at each site,

$$Z(S, T)|Z(S, T-1) = \varphi Z(S, T-1) + E(S, T)$$

  where $E(S, T)$ are independent over time
- A spatial extension allows $E(S, T)$ to be spatially correlated

Spatial AR1 at two sites with $\varphi = 0.95$

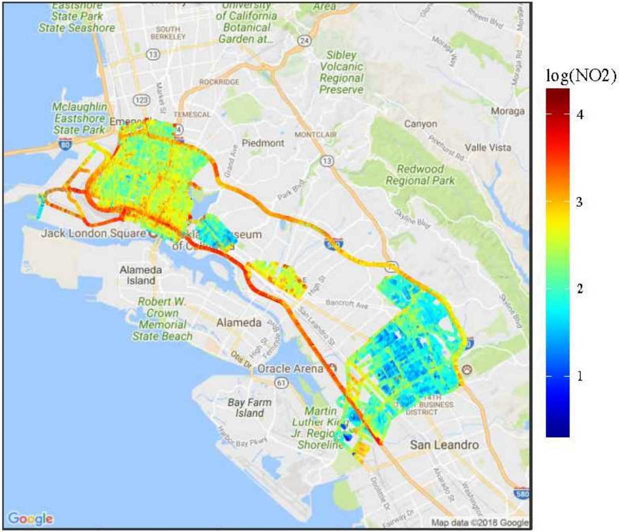


Separable covariance function

- The models so far require repeat observations at a fixed set of sites
- If this is not the case, we need a new approach
- Example: a Google StreetView car drives around a city taking an air pollution measurement every second

Google car data

Median log(NO2), 7/14/2015 - 5/13/2016



Separable covariance function

- In this “irregular” sampling case it is useful to define a spatiotemporal covariance function
- Back to the Z_i notation, there are two distances between the locations of Z_i and Z_j
- The spatial distance is $d_{ij} = ||\mathbf{s}_i - \mathbf{s}_j||$
- The temporal distance is $h_{ij} = |t_i - t_j||$
- Covariance should decrease with both d_{ij} and h_{ij}

Separable covariance function

- A separable covariance function is the product of a spatial and a temporal correlation function

$$\text{Cov}(Z_i, Z_j) = \sigma^2 \rho(d_{ij}) \psi(h_{ij})$$

- Example: exponential spatial correlation $\rho(d_{ij}) = \exp(-d_{ij}/\phi)$
- Example: AR1 temporal correlation $\psi(d_{ij}) = \varphi^{d_{ij}}$
- Fact: If ρ and ψ are valid, the separable covariance is valid
- MLE/MCMC apply as before except with one additional parameter, φ

Spatiotemporal prediction

- Kriging is the optimal (BLUP) spatial prediction
- The prediction is a linear combination of the observations with weights given by the spatial covariance
- The Kriging formula applies for spatiotemporal prediction
- You simply apply the same formula except with a spatiotemporal correlation instead of a spatial correlation

Dynamic linear model (DLM)

- This resembles the spatially-varying coefficients, expect the covariate effect evolve over time

- The model is

$$E(Y_i) = \beta_o(t_i) + \sum_{j=1}^{p} X_{ij}\beta_j(t_i)$$

- The slopes can be given an autoregressive prior

- Example: The relationship between social mobility (X) and COVID-19 infection rate (Y) changes over time as more people begin to wear masks

Software options

- The `spBayes` function `spDynLM` fits a dynamic linear model
- The package `gstat` computes the spatiotemporal variogram/acf and fits parameters by comparing a separable model to these variograms
- The package `spTimer` fit a spatiotemporal AR1 model using MCMC