Spatial point pattern models

Applied Spatial Statistics

Spatial point pattern models

Statistical models for point pattern data have many uses:

- Predicting the location of the next event
- Testing for covariate effects
- Estimating the spatial range of interactions

Spatial point pattern models

We will study models that capture all types of interactions we have discussed:

- Homogeneous Poisson process
- Inhomogeneous Poisson process
- Inhomogeneous Poisson process with covariates
- Strauss process for inhibition
- Cluster process

Poisson process

- Let $\lambda(\mathbf{s})$ be the sampling intensity at location $\mathbf{s} \in \mathcal{D}$
- The expected number of observation in $\mathcal{B} \subset \mathcal{D}$ is the volume under the intensity function

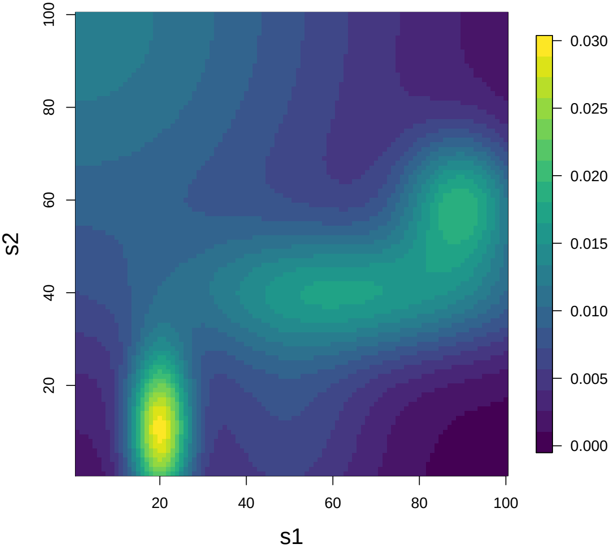
$$\lambda(\mathcal{B}) = \int_{\mathcal{B}} \lambda(\mathbf{s}) d\mathbf{s}$$

- The probability density function (PDF) is

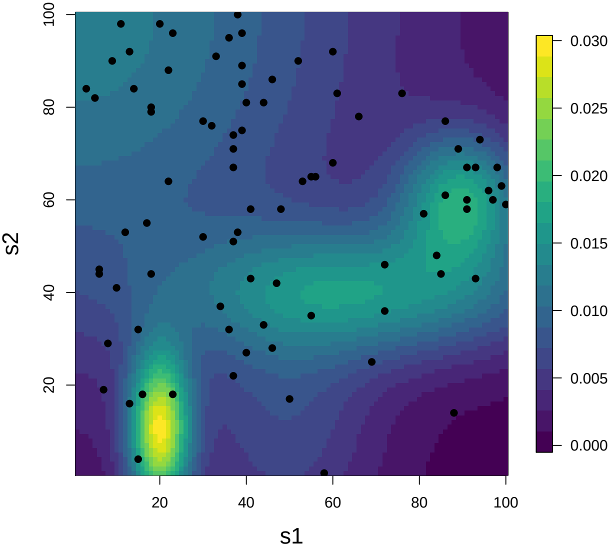
$$f(s) = \frac{1}{c} \lambda(\mathbf{s})$$

where $c = \int_{\mathcal{D}} \lambda(\mathbf{s}) d\mathbf{s}$ is the normalizing constant

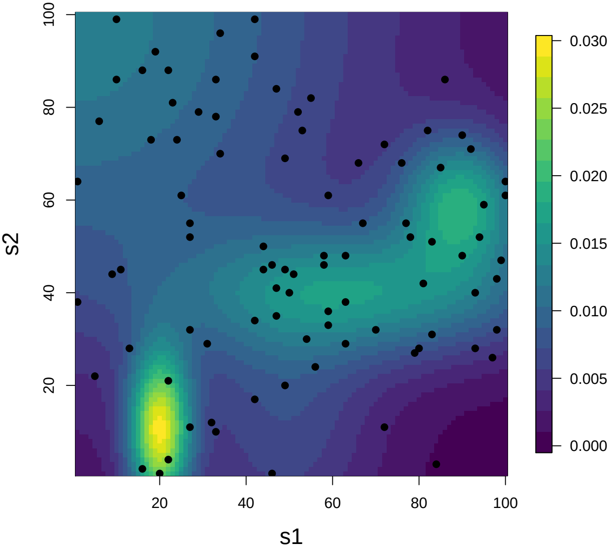
Poisson process intensity function, $\lambda(\mathbf{s})$



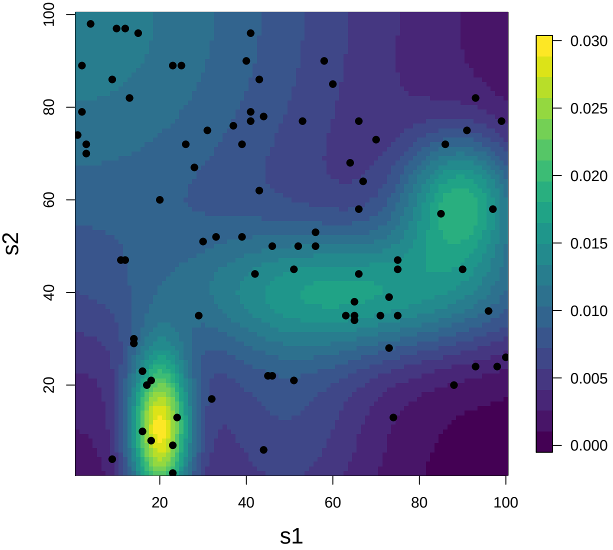
Random sample 1



Random sample 2



Random sample 3



Poisson process

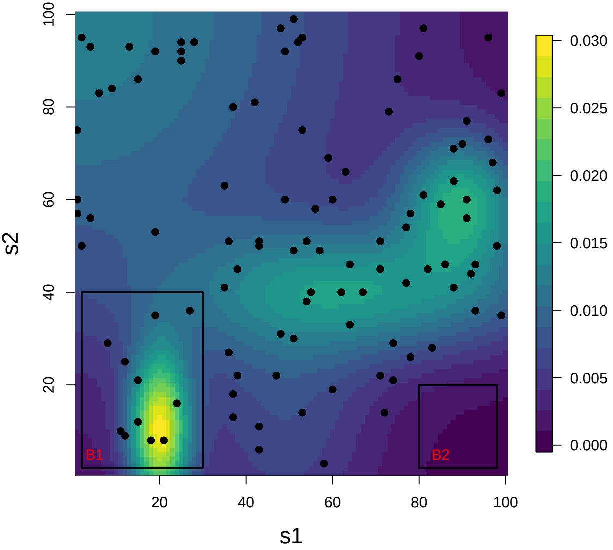
A spatial point pattern is a Poisson process with intensity function $\lambda(\mathbf{s})$ if:

1. The number of samples in $\mathcal{B}$, $Y(\mathcal{B})$, is distributed

$$Y(\mathcal{B}) \sim \text{Poisson}\{\lambda(\mathcal{B})\}$$

2. If $\mathcal{B}_1$ and $\mathcal{B}_2$ are disjoint, then $Y(\mathcal{B}_1)$ and $Y(\mathcal{B}_2)$ are independent

Random sample 4



Homogeneous Poisson process (HPP)

- An HPP has constant intensity, $\lambda(\mathbf{s}) = \lambda_0$
- Therefore, expected counts are proportional to area

$$\lambda(\mathcal{B}) = \int_{\mathcal{B}} \lambda(\mathbf{s}) d\mathbf{s} = |\mathcal{B}|\lambda_0$$

- The PDF is uniform,

$$f(\mathbf{s}) = \frac{\lambda(\mathbf{s})}{\int_{\mathcal{D}} \lambda(\mathbf{s}) d\mathbf{s}} = \frac{1}{|\mathcal{D}|}$$

- Therefore, HPP is completely random sampling

Homogeneous Poisson process (HPP)

Steps to sample from an HPP:

- Draw $n \sim \text{Poisson}\{\lambda(\mathcal{D})\}$ where $\lambda(\mathcal{D}) = \lambda_0|\mathcal{D}|$

- Sample $\mathbf{s}_1, ..., \mathbf{s}_n$ uniformly and independently on $\mathcal{D}$

Inhomogeneous Poisson process (IPP)

- An IPP has spatially-varying intensity $\lambda(\mathbf{s})$
- The intensity function can be modeled similar to the mean function in geostatistics
- A parametric model regresses $\lambda(\mathbf{s})$ onto covariates
- A log-Gaussian Cox process assumes $\log\{\lambda(\mathbf{s})\}$ is a Gaussian process
- Kernel smoothing is nonparametric method to estimate $\lambda(\mathbf{s})$

IPP - Kernel density estimator (KDE)

- Let $\lambda(\mathbf{s}) = f(\mathbf{s})/c$ where $f(\mathbf{s})$ is the PDF and $\int_{\mathcal{D}} f(\mathbf{s})d\mathbf{s} = 1$
- Taking $c = 1/n$ gives $\hat{\lambda}(\mathcal{D}) = n$
- Any density estimator can be used to estimate p
- Simple: partition $\mathcal{D}$ into sub-regions and use the sample proportions in each sub-region to estimate f
- KDE is a smoother version of this,

$$\hat{f}(\mathbf{s}_0) \propto \sum_{i=1}^{n} k(\mathbf{s}_0, \mathbf{s}_i)$$

where k is a kernel function, e.g., $k(\mathbf{s}_i, \mathbf{s}_j) = \exp(-\phi d_{ij}^2)$

Inhomogeneous Poisson process with covariates

- Assume we have p spatial covariates $X_1(\mathbf{s}), ..., X_p(\mathbf{s})$
- Example: $X_j(\mathbf{s})$ is the latitude of $\mathbf{s}$
- Example: $X_j(\mathbf{s})$ is the elevation of $\mathbf{s}$
- Example: $X_j(\mathbf{s})$ is the distance from $\mathbf{s}$ to the coast
- The regression model is $\log\{\lambda(\mathbf{s})\} = \beta_0 + \sum_{j=1}^{p} X_j(\mathbf{s})\beta_j$
- The coefficients $\boldsymbol{\beta} = (\beta_0, ..., \beta_p)$ are interpreted just like Poisson regression

Inhomogeneous Poisson process with covariates

- The parameters can be estimated using MLE
- The conditional (given n) likelihood is

$$l(\boldsymbol{\beta}) = \prod_{i=1}^{n} \frac{\lambda(\mathbf{s}_i; \boldsymbol{\beta})}{\int_{\mathcal{D}} \lambda(\mathbf{s}; \boldsymbol{\beta}) d\mathbf{s}}$$

  where $\lambda(\mathbf{s}; \boldsymbol{\beta}) = \exp\{\sum_{j=1}^{p} X_j(\mathbf{s})\beta_j\}$

- The integral is a problem and needs to be approximated
- The likelihood requires the covariates at all locations in $\mathcal{D}$, not just the n sample locations

Strauss process for inhibition

- A Strauss process discourages pairs of observation to be close to each other
- Let $p_r(\mathbf{s}_1, ..., \mathbf{s}_n)$ be the number of pair of points within r of each other
- The joint PDF of the n sample location is

$$f(\mathbf{s}_1, ..., \mathbf{s}_n) \propto \exp\{-\beta p_r(\mathbf{s}_1, ..., \mathbf{s}_n)\}$$

- The model (without covariates) has two parameters: interaction radius r and repulsion strength β

Strauss process for inhibition

- Estimating β and *r* informs us about interactions
- Example: study effects of social distancing by comparing estimates of *r* and β before and after COVID-19
- Hard-core Strauss process: if $\beta = \infty$ then observations within *r* of each other are strictly prohibited
- Soft-core Strauss process: if $\beta < \infty$ then observations within *r* are discouraged
- CRS: if $\beta = 0$ then observations are independent
- Parameter estimate is difficult because the likelihood has a complicated form

Cluster process

- A Poisson cluster process is a way to model attraction between events
- Below is a simple model, but there are others
- Let the parents $\mathbf{u}_1, ..., \mathbf{u}_K$ be a sample from an HPP
- Parent k gives birth to $m_k \sim$ Poisson children
- The children of parent k are distributed as

$$\mathbf{s}_i \sim \text{Normal}(\mathbf{u}_k, \sigma^2 \mathbf{I}_2)$$