NC STATE UNIVERSITY

# Conquer Big Spatial Data: The Stochastic PDE Approach

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# Motivation

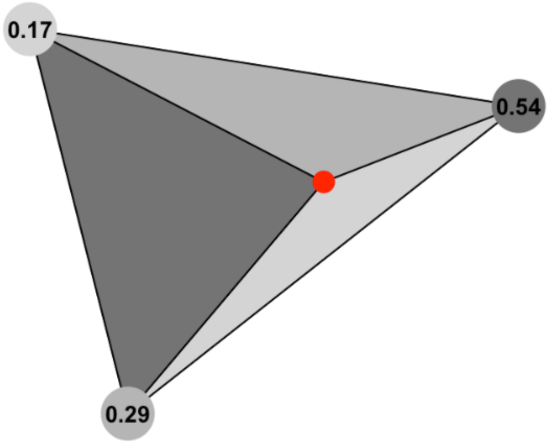
- The method of approximating a continuous Gaussian field using GMRFs was theoretically good but less practical. The SPDE method represents a Gaussian field with Matern covariance by representing a solution of stochastic partial differential equations as a GMRF using the finite element method.
- The GMRF representation of the Gaussian field, which can be computed explicitly, provides a sparse representation of the spatial effect through a sparse precision matrix. This enables the nice computational properties of the GMRFs which can then be implemented in the INLA package.

# The SPDE Model

- $\mathbf{y} \mid \beta_0, \mathbf{u}, \sigma_e^2 \sim N(\beta_0 + \mathbf{Au}, \sigma_e^2)$
- $\mathbf{u} \sim GF(0,\Sigma)$
- $y_i$ are observations at location $s_i$.
- $\beta_0$ is the intercept
- **u** is a spatial Gaussian random field with mean zero and standard deviation Σ
- **A** is the projector matrix

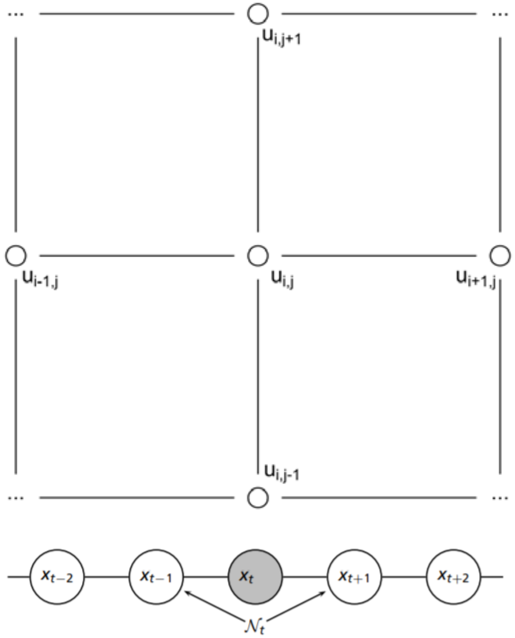
# Mesh

- Points are often distributed irregularly. Need to construct mesh
- Mesh is used in the Finite Element Method to provide a solution to a SPDE.
- To better approximate the field, need to have more triangles.
- Shiny application within INLA package: meshbuilder()



# Understanding the Method

- A Gaussian field with a generalized covariance function obtained in the Matérn correlation function when v>0 is a solution to a SPDE.
- Consider a regular two-dimensional lattice with number of sites tending to infinity.
- The right figure is an example of a GMRF. Another example is AR(1) process
- The GMRF representation is a convolution of processes with precision matrix for distinct values of smoothness v
- As the smoothness parameter v increases, the precision matrix in the GMRF representation becomes denser. This is because the conditional distributions depend on a wider neighborhood.

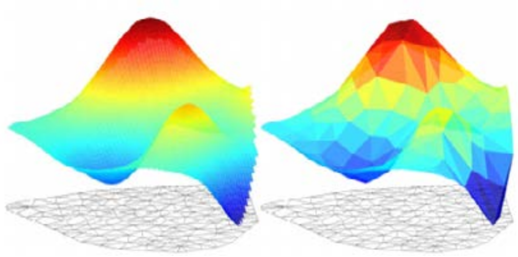


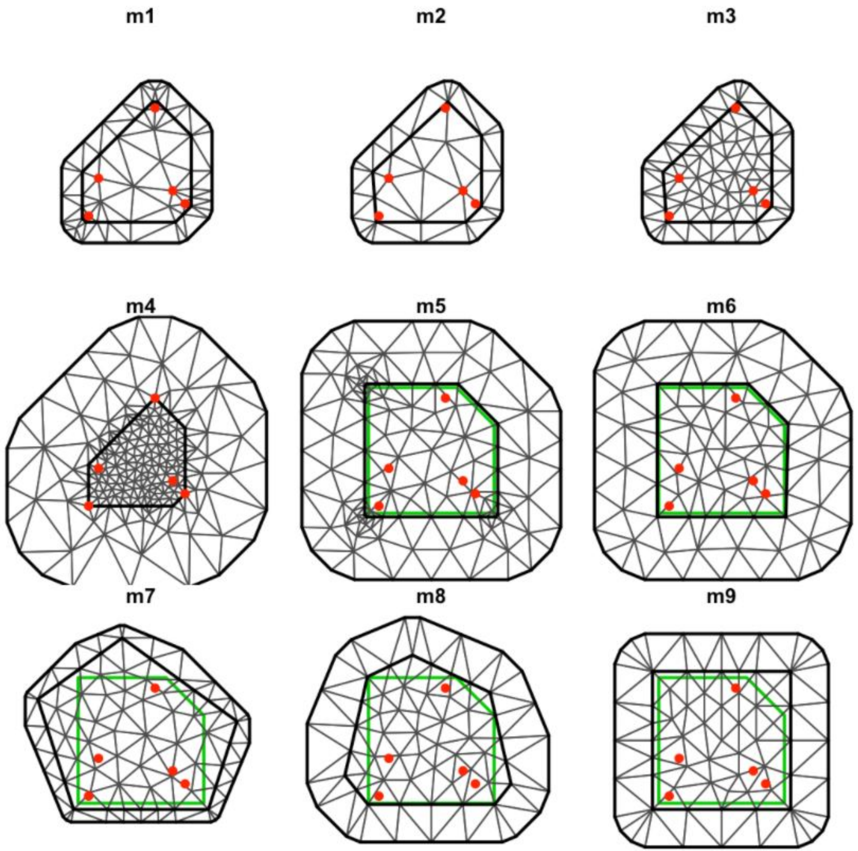
# How Approximation Works

- Illustrate the approximation in two-dimensional space considering piecewise linear basis functions.

  $u(s) = \sum_{j=1}^{m} \psi_k(s) W_k$

- where $\psi_k$ are basis functions and $W_k$ are Gaussian distributed weights, k=1,...,m with m the number of vertices in the triangulation.
- Carefully choose the basis functions to preserve the sparse structure of the resulting precision matrix for the random field at a set of mesh nodes.
- This provides an explicit link between a continuous random field and a GMRF representation, which allows efficient computations.






# To Build a 2-D Mesh

## INLA package in R

### inla.mesh.2d() function

Mandatory arguments:

- *loc*, *loc.domain*, or *boundary*
- *max.edge*: inner domain,

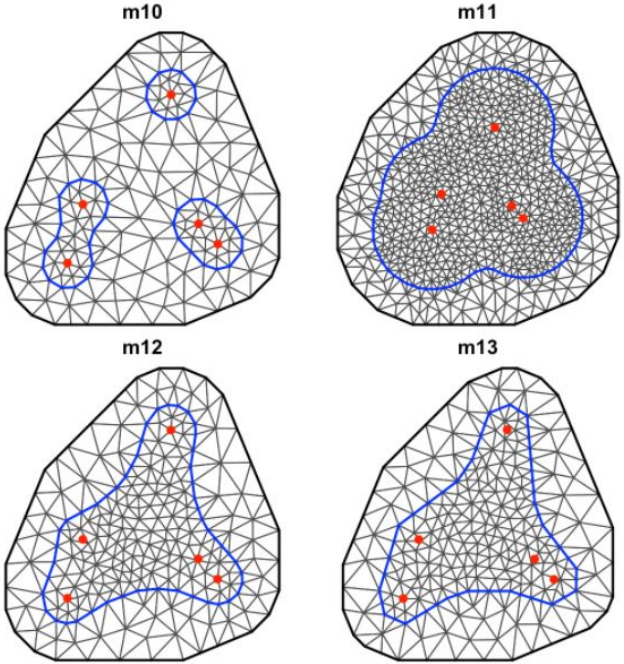
  outer extension

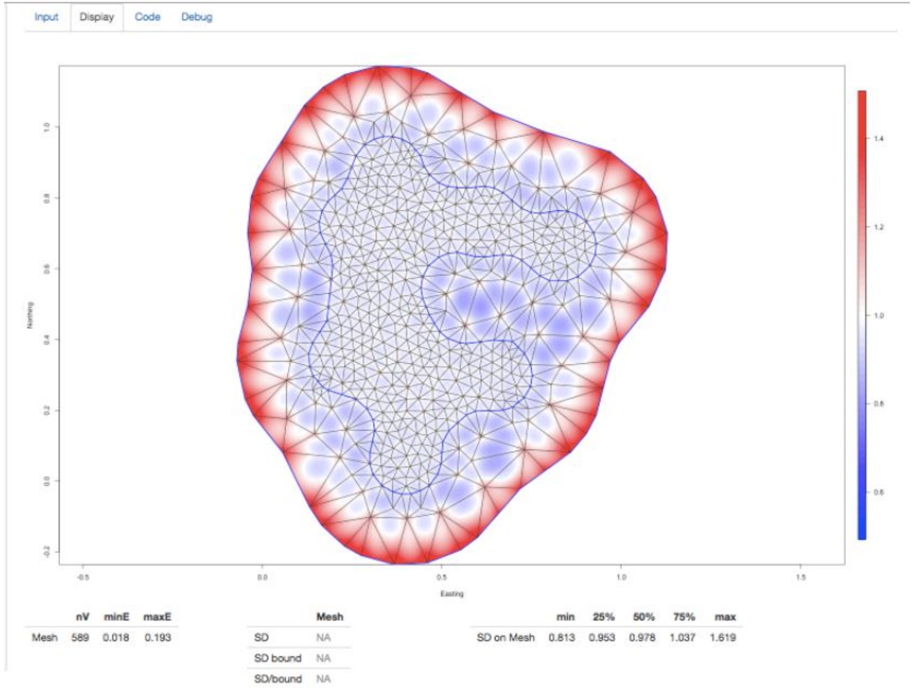
Optional arguments:

- *cutoff*
- *offset*
- *min.angle*
- *n*

* Image is taken from https://becarioprecario.bitbucket.io/spde-gitbook/index.html

# Non-convex Hull Meshes




- boundary <- **inla.nonconvex.hull(***points = , convex = , concave = , resolution = ,…***)**

**R Shiny app**: **meshbuilder()**;

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# Quality of A Mesh

- **Goal**: Uniform triangle shape and size

```
mesh <- inla.mesh.2d( loc = coordinates,
                      max.edge = c(<inner domain>, <outer extension>),
                      cutoff = <a numeric value>,
                      offset = -0.10 <default> )
```

- To create the projector matrix (**A** matrix) from the mesh:

```
A <- inla.spde.make.A(mesh, loc = coordinates)
```

# Problem

- Distance is too small:

  Range(longitude) = 0.44 degrees, range(latitude) = 0.21 degrees

- Re-scale the coordinates:

  (longitude + 96) ×10; (latitude - 41) ×10

- Number of nodes for n = 20,000:

  i. node = 3,312: max.edge = c(0.1, 1), cutoff = 0.05

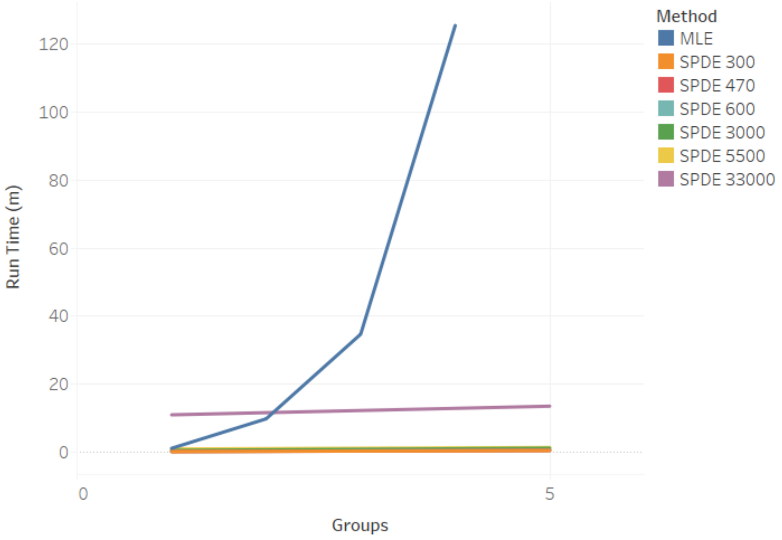
  ii. node = 5,962: max.edge = c(0.08, 0.8), cutoff = 0.03

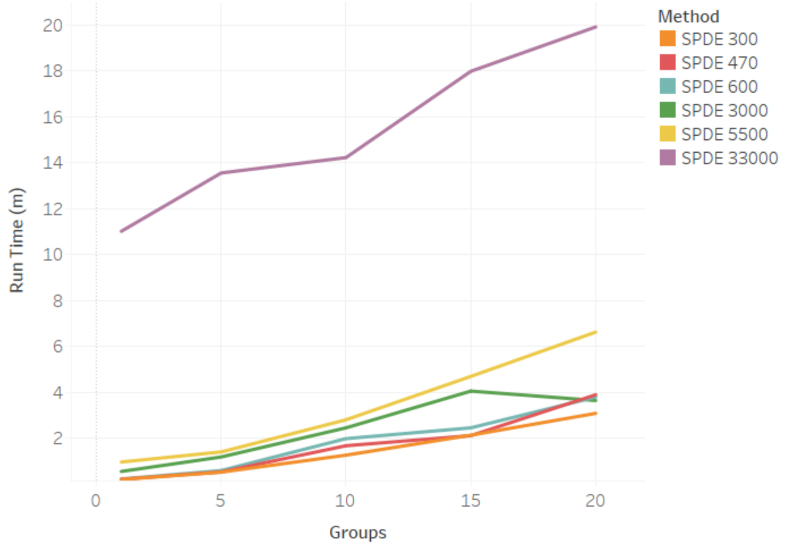
  iii. node = 34,861: max.edge = c(0.03, 0.7), cutoff = 0.01

# The SPDE Model Construction

- $\mathbf{y} \mid \beta_0, \mathbf{u}, \sigma_e^2 \sim N(\beta_0 + \mathbf{A}\mathbf{u}, \sigma_e^2)$
- $\mathbf{u} \sim GF(0, \Sigma)$
- Matern covariance with Penalized Complexity prior: **inla.spde2.pcmatern()**
  i. $\alpha \in [1, 2]; \alpha = \upsilon + {}^{d}/_{2} = 2$ <default>
  ii. $P(\sigma > \sigma_0) = p \Rightarrow P(\sigma > 1) = 0.01$:
     prior.sigma = c(1, 0.01)
  iii. $P(r < r_0) = p \Rightarrow P(r < 1.3) = 0.5$:
     prior.range = c(1.3, 0.5)

# MLE & SPDE Run Time




- MLE Estimation passed over one hour on fourth group
- SPDE Models runtime did not increase exponentially
- SPDE Model with most nodes (~33,000) stayed under half hour with all groups

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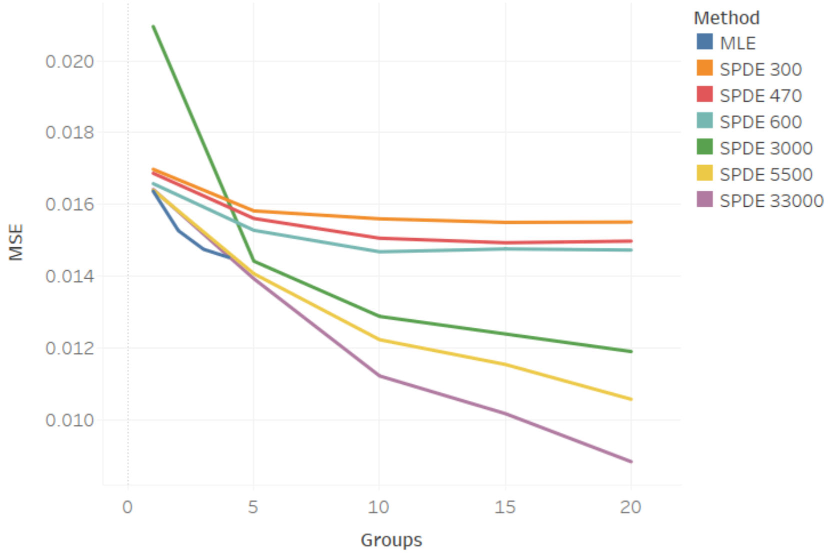
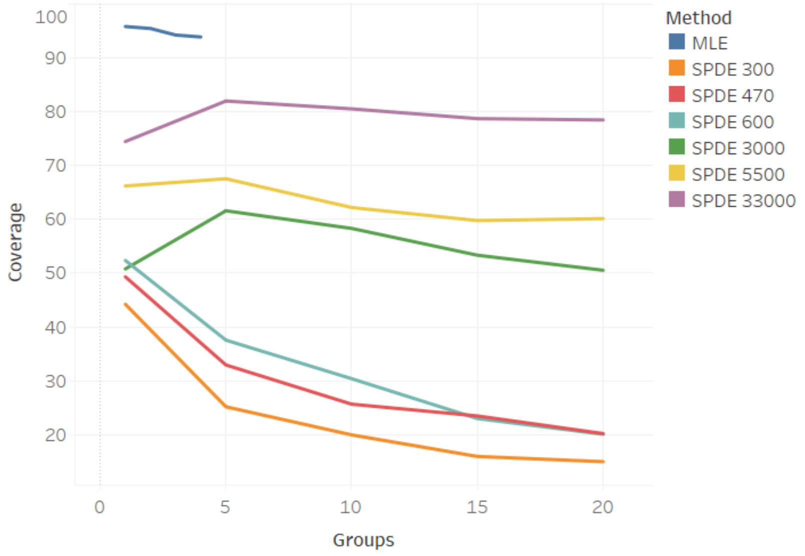
# Performance Across Group Size




- Kriging using MLE Estimation & SPDE showed a decline in MSE as size increased
  - SPDE models with higher node counts decreased more significantly than SPDE models with smaller node counts
- Coverage stayed relatively consistent in MLE and SPDE estimation when node count was high (3000, 5500, 33000) but decreased in coverage as size increased when node counts were small

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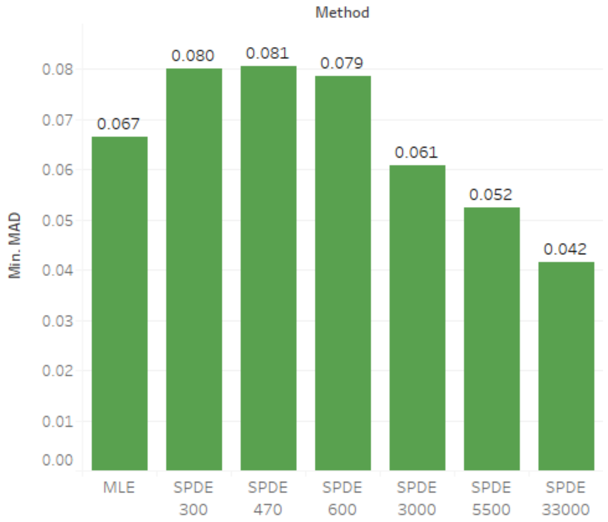
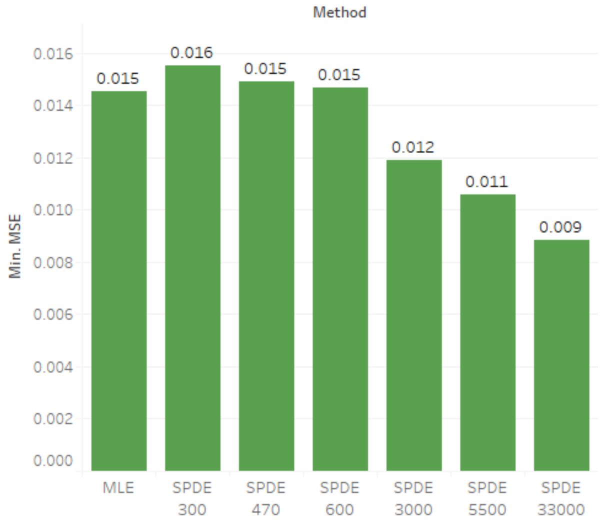
# MAD & MSE




- Kriging methods using SPDE estimation with node size >3,000 performed better than MLE models
- Conversely SPDE estimation with node sizes 300, 470 and 600 performed worse than the MLE estimation

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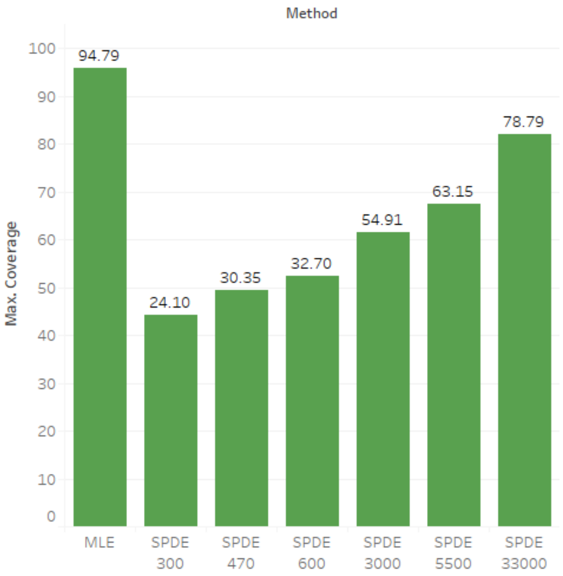
# Coverage vs. Correlation





- Kriging using MLE Estimation had the highest coverage with ~95%
- As the number of nodes increased, coverage increased for SPDE-based Kriging methods
- SPDE methods using nodes 3,000, 5,500 and 33,000 resulted in higher levels of correlation than the MLE estimation

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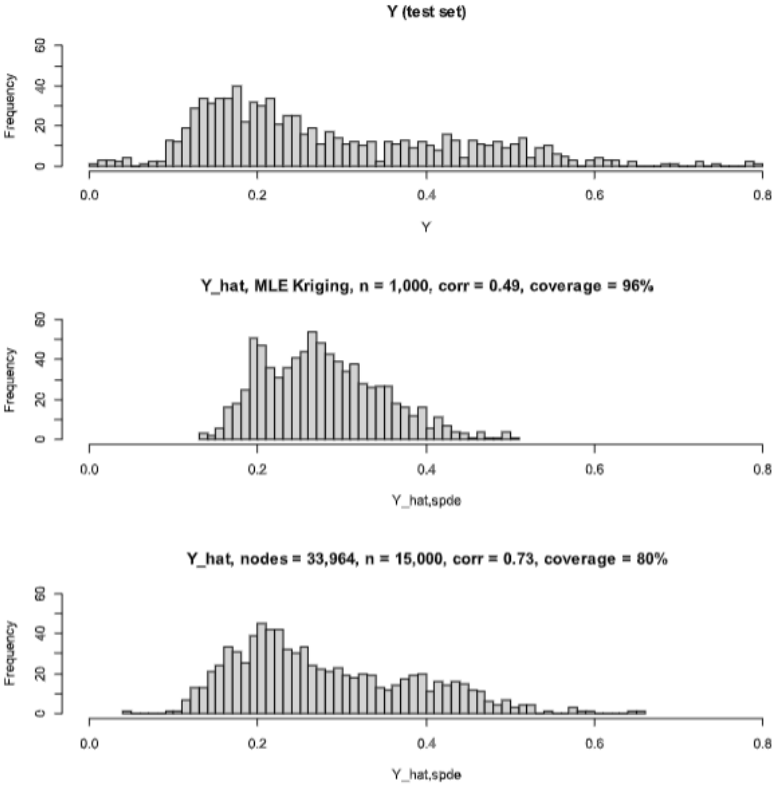
# Methods Overview




- The predictions made with Kriging using the SPDE estimation better correlate with the test set.
- Additionally, it better matches the test data set in terms of the histogram, with more of the tails captured
- The predictions made with Kriging using the MLE estimation do not capture the tails of the test set and are more centered around the central value